Generalised e-Harish-Chandra theory and generic weights for finite reductive groups

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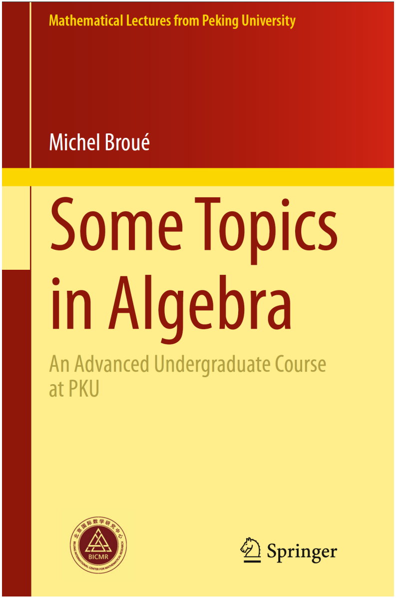
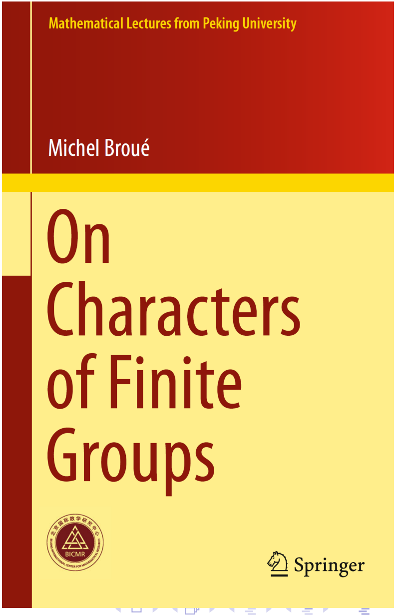
Based on a joint work with Gunter Malle and Jiping Zhang

Representations of Finite Groups and Ramifications
in honor of Michel Broué's 80th birthday
Shenzhen, 28 May 2026

Mathematical Lectures from Peking University





Contents

Harish-Chandra theory

- Let $\mathbf{G}$ be a connected reductive algebraic group over an algebraic closure of a finite field of characteristic p and $F : \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism, some power of which is a Frobenius endomorphism defining a rational structure on $\mathbf{G}$ over $\mathbb{F}_q$, where q is a power of p.
- The group of fixed points $\mathbf{G}^F$ is finite.
- Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in an F-**stable** parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Such Levi subgroup is called $\mathbf{G}$-*split*. We have the Levi decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ and $\mathbf{P}^F = \mathbf{U}^F \rtimes \mathbf{L}^F$, where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$.

Harish-Chandra theory

Let K be a large enough field such that the defining characteristic of $\mathbf{G}$ is invertible in K. The *Harish-Chandra* (or *parabolic*) *induction* and *restriction* functors are respectively

$$\begin{aligned} \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : K\mathbf{L}^F\text{-mod} &\to K\mathbf{G}^F\text{-mod} \\ X &\mapsto K\mathbf{G}^F/\mathbf{U}^F \otimes_{K\mathbf{L}^F} X, \\ {}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : K\mathbf{G}^F\text{-mod} &\to K\mathbf{L}^F\text{-mod} \\ Y &\mapsto \mathrm{Hom}_{K\mathbf{G}^F}(K\mathbf{G}^F/\mathbf{U}^F, Y). \end{aligned}$$

Remark: $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ are independent of the choice of $\mathbf{P}$.

Harish-Chandra theory

- For any character ζ of $\mathbf{L}^F$, the Harish-Chandra induction of ζ to $\mathbf{G}^F$ is

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\zeta) := \mathrm{Ind}_{\mathbf{P}^F}^{\mathbf{G}^F}(\mathrm{Inf}_{\mathbf{L}^F}^{\mathbf{P}^F}(\zeta)),$$

  where Ind and Inf denote respectively the ordinary induction and inflection of characters.
- $\langle \mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\zeta), \chi \rangle_{\mathbf{G}^F} = \langle \zeta, {}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\chi) \rangle_{\mathbf{L}^F}$ for any character ζ of $\mathbf{L}^F$ and any character χ of $\mathbf{G}^F$.

Harish-Chandra theory

- For a finite group G, let $\mathrm{Irr}(G)$ denote the set of irreducible (complex) character of G.
- An irreducible character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ is called *cuspidal* if ${}^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\chi) = 0$ for any proper **G**-split Levi subgroup **L** of **G**.
- A cuspidal pair of **G** is a pair $(\mathbf{L}, \zeta)$, where **L** is a **G**-split Levi subgroup **L** of **G** and $\zeta \in \mathrm{Irr}(\mathbf{L}^F)$ is a cuspidal character of $\mathbf{L}^F$.
- For any cuspidal pair $(\mathbf{L}, \zeta)$ of **G**, we define the *Harish-Chandra series* $\mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}, \zeta))$ to be the set consisting of the irreducible constituents of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\zeta)$.

Harish-Chandra theory

Theorem (Harish-Chandra theory)

1. *Let* $(\mathbf{L}, \zeta)$ *and* $(\mathbf{L}', \zeta')$ *be two cuspidal pairs of* $\mathbf{G}$. *Then*

$$\mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}, \zeta)) \cap \mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}', \zeta')) \neq \emptyset$$

if and only if $(\mathbf{L}, \zeta)$ *and* $(\mathbf{L}', \zeta')$ *are* $\mathbf{G}^F$*-conjugate.*

2. *The irreducible representations of* $\mathbf{G}^F$ *fall into Harish-Chandra series*

$$\mathrm{Irr}(\mathbf{G}^F) = \coprod_{(\mathbf{L}, \zeta)} \mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}, \zeta)).$$

where $(\mathbf{L}, \zeta)$ *runs through the* $\mathbf{G}^F$*-conjugacy classes of cuspidal pairs of* $\mathbf{G}$.

Harish-Chandra theory

1. Let $(\mathbf{L}, \zeta)$ be a cuspidal pair of $\mathbf{G}$.
2. The characters in $\mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}, \zeta))$ are in bijection with the irreducible representations of

$$\mathscr{H}_{\mathbf{G}}(\mathbf{L}, \zeta) := \mathrm{End}_{K\mathbf{G}^F}(\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(X))^{opp},$$

   where X is a $K\mathbf{L}^F$-module affording ζ.
   This is the *Hecke algebra* associated with $(\mathbf{L}, \zeta)$.
3. Write $W_{\mathbf{G}^F}(\mathbf{L}, \zeta) := \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})_\zeta / \mathbf{L}^F$ for the relative Weyl group.
4. Let $K = \mathbb{C}$. Then $\mathscr{H}_{\mathbf{G}}(\mathbf{L}, \zeta) \simeq KW_{\mathbf{G}^F}(\mathbf{L}, \zeta)$, which yields a bijection

$$I_{\mathbf{L},\zeta}^{\mathbf{G}} : \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}, \zeta)) \to \mathscr{E}(\mathbf{G}^F \mid (\mathbf{L}, \zeta)).$$

Theorem (Howlett–Lehrer Comparison Theorem)

Let $(\mathbf{L},\zeta)$ be a cuspidal pair in $\mathbf{G}$. Then the collection of bijections

$$I^{\mathbf{M}}_{\mathbf{L},\zeta} : \mathrm{Irr}(W_{\mathbf{M}^F}(\mathbf{L},\zeta)) \longrightarrow \mathscr{E}(\mathbf{M}^F \mid (\mathbf{L},\zeta)),$$

where $\mathbf{M}$ runs over $\mathbf{G}$-split Levi subgroups $\mathbf{L} \leq \mathbf{M} \leq \mathbf{G}$, can be chosen such that the diagrams

$$\begin{array}{ccc} \mathbb{Z}\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\zeta)) & \xrightarrow{I^{\mathbf{G}}_{\mathbf{L},\zeta}} & \mathbb{Z}\mathscr{E}(\mathbf{G}^F \mid (\mathbf{L},\zeta)) \\ \uparrow\mathrm{Ind} & & \uparrow\mathrm{R}^{\mathbf{G}}_{\mathbf{M}} \\ \mathbb{Z}\mathrm{Irr}(W_{\mathbf{M}^F}(\mathbf{L},\zeta)) & \xrightarrow{I^{\mathbf{M}}_{\mathbf{L},\zeta}} & \mathbb{Z}\mathscr{E}(\mathbf{M}^F \mid (\mathbf{L},\zeta)) \end{array}$$

commute for all $\mathbf{M}$.

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Sylow e-theory

For $e \geq 1$, let $\phi_e(x) \in \mathbb{Z}[x]$ be the eth cyclotomic polynomial over $\mathbb{Q}$.
The *generic* (or *polynomial*) *order* of $(\mathbf{G}, F)$ is a monic polynomial in $\mathbb{Z}[x]$:

$$|\mathbb{G}| = x^{|\Phi^+|} \prod_e \phi_e(x)^{a_e}$$

such that $|\mathbf{G}^{F^m}| = |\mathbb{G}|(q^m)$ for any positive integer m.

Sylow e-theory

G^F	$\lvert G^F \rvert$
$SL_n(q)$	$q^{\frac{n(n-1)}{2}}(q^2-1)(q^3-1)\cdots(q^n-1)$
$SU_n(q)$	$q^{\frac{n(n-1)}{2}}(q^2-1)(q^3+1)\cdots(q^n-(-1)^n)$
$SO_{2n+1}(q)$	$q^{n^2}(q^2-1)(q^4-1)\cdots(q^{2n}-1)$
$Sp_{2n}(q)$	$q^{n^2}(q^2-1)(q^4-1)\cdots(q^{2n}-1)$
$SO_{2n}^+(q)$	$q^{n^2-n}(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n-1)$
$SO_{2n}^-(q)$	$q^{n^2-n}(q^2-1)(q^4-1)\cdots(q^{2n-2}-1)(q^n+1)$
$G_2(q)$	$q^6(q^2-1)(q^6-1)$
${}^3D_4(q)$	$q^{12}(q^2-1)(q^6-1)(q^8+q^4+1)$
$F_4(q)$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$E_6(q)$	$q^{36}(q^2-1)(q^5-1)(q^6-1)(q^8-1)(q^9-1)(q^{12}-1)$
${}^2E_6(q)$	$q^{36}(q^2-1)(q^5+1)(q^6-1)(q^8-1)(q^9+1)(q^{12}-1)$
$E_7(q)$	$q^{63}(q^2-1)(q^6-1)(q^8-1)(q^{10}-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)$
$E_8(q)$	$q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)$

Math. Ann. 292, 241–262 (1992)

Mathematische Annalen

Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis*

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Generic Sylow Theorems

- An F-stable torus $\mathbf{T} \leq \mathbf{G}$ is called a *e-torus* if its generic order equals $\phi_e(x)^a$ for some $a \geq 0$.
- We call $\mathbf{T}$ a *Sylow e-torus* of $\mathbf{G}$ if its generic order $\phi_e(x)^{a_e}$ is the precise power of $\phi_e(x)$ dividing the generic order of $\mathbf{G}$.
- Clearly, e-tori of $(\mathbf{G}, F)$ become 1-tori in $(\mathbf{G}, F^e)$.

Theorem (Broué–Malle, 1992)

(a) *There exist Sylow e-tori in $\mathbf{G}$, and they are all $\mathbf{G}^F$-conjugate.*

(b) *Any e-torus of $\mathbf{G}$ is contained in some Sylow e-torus.*

- For any F-stable torus $\mathbf{T}$, denote by $\mathbf{T}_{\phi_e}$ its Sylow e-torus.

e-Harish-Chandra theory

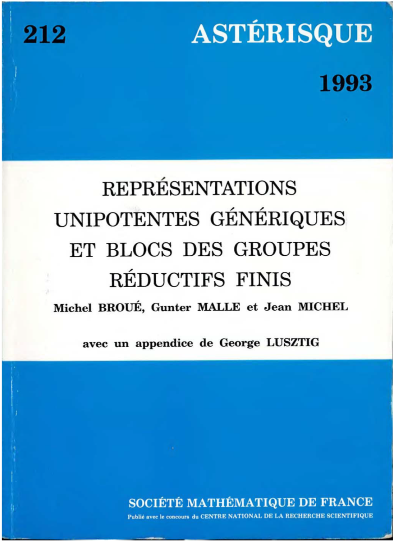
- An *e-split Levi subgroup* $\mathbf{L}$ of $\mathbf{G}$ is the centralizer of some e-torus of $\mathbf{G}$. In this case, $\mathbf{L} = \mathrm{C}_{\mathbf{G}}(\mathrm{Z}^{\circ}(\mathbf{L})_{\phi_e})$.
- $\mathbf{G}$-split Levi subgroups are exactly 1-split Levi subgroups.
- We can define e-cuspidal characters, e-cuspidal pairs, e-Harish-Chandra series analogously as cuspidal characters, cuspidal pairs, Harish-Chandra series, by replacing $\mathbf{G}$-split Levi subgroups, Harish-Chandra induction and restriction with e-split Levi subgroups, Lusztig induction and restriction respectively.

Example:

- For the group $\mathbf{G}^F = \mathrm{GL}_n(q)$, the e-split Levi subgroups $\mathbf{L}$ are of rational form:

$$\mathbf{L}^F = \mathrm{GL}_{n_1}(q^e) \times \cdots \times \mathrm{GL}_{n_s}(q^e) \times \mathrm{GL}_s(q).$$

- Write $n = ae + r$ with $0 \leq r < e$. A Sylow e-torus of GL_n has centraliser of rational form $\mathrm{GL}_1(q^e)^a \times \mathrm{GL}_r(q)$.
- Unipotent e-cuspidal characters ζ of $\mathbf{G}^F = \mathrm{GL}_n(q)$ are labelled by e-core partitions.



GENERIC BLOCKS OF FINITE REDUCTIVE GROUPS

Michel Broué, Gunter Malle and Jean Michel

Ecole Normale Supérieure and Universität Heidelberg

To Charlie Curtis

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1991 *Mathematics Subject Classification.* 20, 20G.

We thank the M.S.R.I. in Berkeley, and the second author thanks the Ecole Normale Supérieure, for their hospitality during the elaboration of part of this work

S. M. F.
Astérisque 212* (1993)

Theorem (Broué–Malle–Michel, 1993)

The unipotent e-Harish-Chandra series partition the set of unipotent characters of $\mathbf{G}^F$*, that is,*

$$\mathrm{Uch}(\mathbf{G}^F) = \coprod_{(\mathbf{L},\zeta)} \mathscr{E}(\mathbf{G}^F, (\mathbf{L}, \zeta)),$$

where the union runs over e-cuspidal unipotent pairs (L, ζ) *in* $\mathbf{G}$ *modulo* $\mathbf{G}^F$*-conjugation.*

Theorem (Comparison Theorem)

For any e-cuspidal unipotent pair $(\mathbf{L}, \zeta)$ *in* $\mathbf{G}$*, there exists a collection of isometries*

$$I_{\mathbf{L},\zeta}^{\mathbf{M}} : \mathbb{Z}\,\mathrm{Irr}(W_{\mathbf{M}^F}(L, \lambda)) \to \mathbb{Z}\mathscr{E}(\mathbf{M}^F, (\mathbf{L}, \zeta)),$$

where $\mathbf{M}$ *runs over e-split Levi subgroups* $\mathbf{L} \leq \mathbf{M} \leq \mathbf{G}$*, such that the diagram*

$$\begin{array}{ccc} \mathbb{Z}\,\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}, \zeta)) & \xrightarrow{I_{\mathbf{L},\zeta}^{\mathbf{G}}} & \mathbb{Z}\mathscr{E}(\mathbf{G}^F, (\mathbf{L}, \zeta)) \\ \uparrow Ind & & \uparrow \mathrm{R}_{\mathbf{M}}^{\mathbf{G}} \\ \mathbb{Z}\,\mathrm{Irr}(W_{\mathbf{M}^F}(\mathbf{L}, \zeta)) & \xrightarrow{I_{\mathbf{L},\zeta}^{\mathbf{M}}} & \mathbb{Z}\mathscr{E}(\mathbf{M}^F, (\mathbf{L}, \zeta)) \end{array}$$

commutes for all $\mathbf{M}$.

Blocks of finite reductive groups

Theorem (Cabanes–Enguehard, 1999)

Suppose that $\ell \nmid q$ is an odd prime that is good for $\mathbf{G}$, and $\ell \neq 3$ if $\mathbf{G}^F$ is of type ${}^3\mathrm{D}_4$. Let e be the multiplicative order of q modulo ℓ. Then the ℓ-blocks of $\mathbf{G}^F$ are in bijection with the $\mathbf{G}^F$-conjugacy classes of e-cuspidal pairs of $\mathbf{G}$.

M. Cabanes, M. Enguehard, On unipotent blocks and their ordinary characters. *Invent. Math.* **117** (1994), 149–164.

M. Cabanes, M. Enguehard, On blocks of finite reductive groups and twisted induction. *Adv. Math.* **145** (1999), 189–229.

R. Kessar, G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture. *Ann. of Math. (2)* **178** (2013), 321–384.

Contents

- Let ℓ be a prime. An *ℓ-weight* of G is a pair (Q, π), where Q is an ℓ-subgroup of G and π is a projective simple $k\,\mathrm{N}_G(Q)/Q$-module.

Denote by $\mathrm{Alp}(G)$ the set of all G-conjugacy classes of weights of G.

The *Alperin weight conjecture*:

Conjecture (Jonathan L. Alperin, 1986)

Let G be a finite group and ℓ be a prime. Then

$$|\operatorname{IBr}(G)| = |\operatorname{Alp}(G)|.$$

Blockwise version

Each weight may be assigned to a unique block. Let B be an ℓ-block of G, and (Q, π) a weight of G. A weight (Q, π) is said to be a B-*weight* of G if $b^G = B$, where b is the block of $\mathrm{N}_G(Q)$ containing π.
Denote the set of G-conjugacy classes of B-weights of G by $\mathrm{Alp}(B)$.

The *blockwise Alperin weight conjecture*:

Conjecture (Alperin, 1986)

Let G be a finite group, ℓ a prime and B be an ℓ-block of G. Then

$$|\,\mathrm{IBr}(B)| = |\,\mathrm{Alp}(B)|.$$

Example: finite groups of Lie type in defining characteristic

- Assume that $\ell \mid q$.
- The group $\mathbf{G}$ has an F-stable maximal torus $\mathbf{T}$ and an F-stable Borel subgroup $\mathbf{B}$ with $\mathbf{T} \subseteq \mathbf{B}$.
- If R is a radical ℓ-subgroup, then up to conjugacy, $R = \mathbf{U}^F$, where $\mathbf{U}$ is the unipotent radical of some parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$. In particular, $\mathbf{P}^F = \mathbf{U}^F \rtimes \mathbf{L}^F$ and $\mathbf{P}^F = \mathrm{N}_{\mathbf{G}^F}(\mathbf{U}^F)$.
- Let $\mathrm{St}_{\mathbf{L}^F}$ be the Steinberg module of $\mathbf{L}^F$, and μ be any one-dimensional module of $\mathbf{L}^F$. Then $\mu\mathrm{St}_{\mathbf{L}^F}$ is a projective simple $k\mathbf{L}^F$module. That is, $(\mathbf{U}^F, \mu\mathrm{St}_{\mathbf{L}^F})$ is an ℓ-weight of $\mathbf{G}^F$.

Example: finite groups of Lie type in defining characteristic

- Each ℓ-weight of $\mathbf{G}^F$ can be constructed in this way. (Alperin, Cabanes, 1986–1988)
- **Local:** Up to conjugacy, the ℓ-weights of $\mathbf{G}^F$ are in bijection with the set of pairs $(\mathbf{P}, \mu)$, where $\mathbf{P}$ is parabolic subgroup of $\mathbf{G}$ containing $\mathbf{B}$ and $\mu \in \mathrm{Hom}(\mathbf{P}^F, k^\times)$.
- **Global:** Up to isomorphism, the simple $k\mathbf{G}^F$-modules V are in bijection with the set of pairs $(\mathbf{P}, \mu)$ defined as above. This bijection is defined as follows (Curtis): every simple $k\mathbf{G}^F$-module V contains a unique $\mathbf{B}^F$-invariant one-dimensional space kv, $\mathbf{P}^F$ is the stabilizer of kv in $\mathbf{G}^F$ and $g(v) = \mu(g)v$ for all $g \in \mathbf{P}^F$.

Reformulations and related developments

- in terms of twisted group algebra and fusion systems (Kessar 2007).
- (Kessar–Malle–Semeraro 2023) for compact Lie groups: fusion system $\longleftrightarrow$ representations of Weyl groups.
- (Kessar–Malle–Semeraro 2023) for connected reductive algebraic groups.

Reduction of Alperin weight conjecture

The Alperin weight conjecture has been reduced to the simple groups.

Theorem (Navarro–Tiep, 2011)

Assume that every non-abelian simple group satisfies the **inductive Alperin weight (AW) condition***. Then the Alperin weight conjecture is true.*

Theorem (Späth, 2013)

Assume that every non-abelian simple group satisfies the **inductive blockwise Alperin weight (BAW) condition***. Then the blockwise Alperin weight conjecture is true.*

G. Navarro, P. H. Tiep, A reduction theorem for the Alperin weight conjecture. *Invent. Math.* **184** (2011), 529–565.

B. Späth, A reduction theorem for the blockwise Alperin weight conjecture. *J. Group Theory* **16** (2013), 159–220.

Verification of the inductive condition

The inductive BAW condition holds for

- alternating groups (Malle, 2014),
- sporadic simple groups (Breuer–F.),
- groups of Lie type and defining characteristic (Späth, 2013).

Thus only the simple groups of Lie type at primes p different from their defining characteristic remain to be considered.
Some cases of small rank were verified:

- Suzuki groups and Ree groups (Malle, 2014),
- groups of Lie type G_2 and ${}^3D_4$ (Schulte, 2016, 56 pages).

Groups of Lie type and non-defining characteristic

The inductive BAW condition holds for

- groups of Lie type F_4 and any odd prime (An–Hiß–Lübeck, Memo. AMS, 150+ pages!!!)
- simple groups of type C_n and the prime 2 (F.–Malle, 2022)
- simple groups of type A_n and ${}^2\mathrm{A}_n$. (F.–C. Li–Zhang, 2021; F.–Z. Li–Zhang, 2023)
- groups of Lie type F_4 and the prime 2 (F.–Yu–Zhang, 2023)

Under the unitriangularity assumption of decomposition matrices, the inductive BAW condition has been proved for

- type C_n and any odd prime, (C. Li, 2021)
- type B_n and any odd prime. (F.–C.Li–Zhang, 2022)

Simplify: reduce to quasi-isolated blocks

The reduction approach for McKay conjecture

Conjecture (McKay 1971)

Let G be a finite group, ℓ be a prime and P be a Sylow ℓ-subgroup. Then $|\operatorname{Irr}_{\ell'}(G)| = |\operatorname{Irr}_{\ell'}(\mathrm{N}_G(P))|$.

Here $\operatorname{Irr}_{\ell'}(G)$ denotes the set of $\chi \in \operatorname{Irr}(G)$ with $\ell \nmid \chi(1)$.

McKay conjecture has been

- reduced to simple groups by Isaacs–Malle–Navarro (2007).
- proved by Malle–Späth (2016) for the prime 2, and Cabanes–Späth (2023) for odd primes, using this reduction.

McKay conjecture

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GROUP THEORY

After 20 Years, Math Couple Solves Major Group Theory Problem

Britta Späth has dedicated her career to proving a single, central conjecture. She's finally succeeded, alongside her partner, Marc Cabanes.

McKay 猜想：群论中的见微知著

张和持

中国科学院数学与系统科学研究院

2025年5月22日 14:53 北京

16人

McKay conjecture

In the proof of the inductive condition of McKay conjecture given by Isaacs–Malle–Navarro, the normalizer of a Sylow e-torus is used, instead of the normalizer of Sylow subgroups.

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Assume that $\ell \nmid q$. Let e be the multiplicative order of q modulo ℓ. Let $E := E_{e,\ell} = \{\, e\ell^i \mid i = 0, 1, 2, \ldots \}$ or $\{1, 2, 4, 8, \ldots\}$.

Definition

Let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$. We say χ is e-generalized-cuspidal (e-GC) if χ is an irreducible constituent of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ where $(\mathbf{L}, \lambda)$ is an E-cuspidal pair of $\mathbf{G}$ with $\mathrm{Z}^\circ(\mathbf{L})_{\phi_e} \subseteq \mathrm{Z}(\mathbf{G})$.

- Let $s \in {\mathbf{G}^*}^F$ be semisimple. We say $\chi \in \mathscr{E}(\mathbf{G}^F, s)$ is *e-Jordan-generalized-cuspidal (e-JGC)* if
  - $\mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}^*}(s))_{\phi_e} = \mathrm{Z}^\circ(\mathbf{G}^*)_{\phi_e}$, and
  - χ corresponds under Jordan decomposition to the $\mathrm{C}_{\mathbf{G}^*}(s)^F$-orbit of a unipotent e-GC character of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)^F$.
- If $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ is e-GC (resp. e-JGC), then $(\mathbf{L}, \lambda)$ is called an *e-GC pair* (resp. *e-JGC pair*) of $\mathbf{G}$.

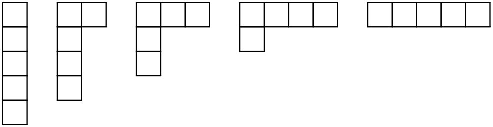
Lemma

Assume that ℓ is odd, good for $\mathbf{G}$, and $\ell > 3$ if $\mathbf{G}^F$ has a component of type ${}^3\mathrm{D}_4$. If $\ell \nmid |\mathrm{Z}(\mathbf{G}_{\mathrm{sc}})^F|$ then $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ is e-GC if and only if χ is e-cuspidal.

e-generalized-cuspidality

Proposition

Let $\mathbf{G} = \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ *(with* $n \geq 2$*) and* $F\colon \mathbf{G} \to \mathbf{G}$ *be a Frobenius endomorphism such that* $\mathbf{G}^F = \mathrm{SL}_n(\epsilon q)$ *with* $\epsilon \in \{\pm 1\}$*. Assume that* ℓ *is odd. Then* $\mathbf{G}^F$ *possesses a unipotent e-GC character that is not e-cuspidal if and only if* $\ell \mid (q - \epsilon)$*, and* $n = \ell^k$ *for some integer* $k \geq 1$*, in which case the unipotent e-GC characters of* $\mathbf{G}^F$ *are those parameterized by the hook partitions* $(n), (n-1, 1), (n-2, 1^2), \ldots, (1^n)$ *of* n*.*



Generic weights

- For an ℓ-block B of $\mathbf{G}^F$, we denote by $\mathcal{L}(B)$ the set of e-JGC pairs $(\mathbf{L}, \lambda)$ of $\mathbf{G}^F$ such that $\lambda \in \mathscr{E}(\mathbf{L}^F, \ell')$ and there is some $\chi \in \mathrm{Irr}(B)$ with $\langle \chi, \mathrm{R}^{\mathbf{G}}_{\mathbf{L} \subseteq \mathbf{P}}(\lambda) \rangle \neq 0$ for any parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{L}$ as a Levi subgroup.

Definition (Generic weights)

Let B be an ℓ-block of $\mathbf{G}^F$. A generic weight of B is a pair $(\mathbf{T}, \eta)$ such that $\mathbf{T} = \mathrm{Z}^\circ(\mathrm{C}_{\mathbf{G}}(\mathbf{T}))_{\phi_e}$ is an e-torus and $\eta \in \mathrm{Irr}(\mathrm{N}_{\mathbf{G}^F}(\mathbf{T}) \mid \lambda)$ such that $(\eta(1)/\lambda(1))_\ell = |\mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\mathrm{C}_{\mathbf{G}^F}(\mathbf{T})|_\ell$, $\lambda \in \mathscr{E}(\mathrm{C}_{\mathbf{G}^F}(\mathbf{T}), \ell')$ with $(\mathrm{C}_{\mathbf{G}}(\mathbf{T}), \lambda) \in \mathcal{L}(B)$.

- Denote by $\mathcal{W}(\mathbf{G}^F)$ the set of conjugacy classes of generic weights of $\mathbf{G}^F$, and by $\mathcal{W}(B)$ the set of conjugacy classes of generic weights of $\mathbf{G}^F$ belonging to B.

Generic weights

Suppose that $\mathbf{G}$ is simple and simply connected and F is a Frobenius endomorphism. Let ℓ be a prime such that ℓ is odd and good for $\mathbf{G}$ with $\ell \nmid q|\mathrm{Z}(\mathbf{G})^F|$. Assume that $\ell > 3$ if $\mathbf{G}^F = {}^3\mathrm{D}_4(q)$. Let $\widetilde{\mathbf{G}}$ be a regular embedding of G and $\mathcal{B}$ consists of field and graph automorphisms.

Theorem (F.–Malle–Zhang)

Let B be an ℓ-block of $\mathbf{G}^F$. Then there is a $(\widetilde{\mathbf{G}}^F \rtimes \mathcal{B})_B$-equivariant bijection $\Omega\colon \mathcal{W}(B) \to \mathrm{Alp}(B)$ such that for every $\overline{(\mathbf{T},\eta)} \in \mathcal{W}(B)$, there exists a B-weight (R,φ) of $\mathbf{G}^F$ with $\overline{(R,\varphi)} = \Omega(\overline{(\mathbf{T},\eta)})$ satisfying

(a) $\mathbf{T} = \mathrm{Z}^\circ(\mathrm{C}^\circ_{\mathbf{G}}(\mathrm{Z}(R)))_{\phi_e}$, $\mathrm{bl}(\varphi)^{\mathrm{N}_{\mathbf{G}^F}(\mathbf{T})} = \mathrm{bl}(\eta)$ *and*

(b) $((\widetilde{\mathbf{G}}^F \rtimes \mathcal{B})_{\mathbf{T},\eta}, \mathrm{N}_{\mathbf{G}^F}(\mathbf{T}), \eta) \geqslant_b ((\widetilde{\mathbf{G}}^F \rtimes \mathcal{B})_{R,\varphi}, \mathrm{N}_{\mathbf{G}^F}(R), \varphi)$.

Z. Feng, G. Malle, J. Zhang, Generic weights for finite reductive groups. *Math. Ann.* **394** (2026), 90.

Assume that $\mathbf{G}$ is connected reductive, F a Frobenius map, and ℓ is odd, good for $\mathbf{G}$ and does not divide $q \cdot |(\mathrm{Z}(\mathbf{G})/\mathrm{Z}^\circ(\mathbf{G}))^F|$.

Question

Let $(\mathbf{L}_0, \zeta_0)$ be an e-Jordan-cuspidal pair of $\mathbf{G}$ with $\zeta_0 \in \mathscr{E}(\mathbf{L}_0^F, \ell')$. Is there a bijection

$$\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L}_0, \zeta_0)) \to \coprod_{(\mathbf{L},\zeta)} \mathrm{dz}(W_{\mathbf{G}^F}(\mathbf{L}, \zeta))$$

where $(\mathbf{L}, \zeta)$ runs through the $\mathbf{G}^F$-conjugacy classes of e-JGC pairs of $\mathbf{G}$ with $\zeta \in \mathscr{E}(\mathbf{L}^F, \ell')$ and $\mathrm{R}_{\mathbf{L}_0}^{\mathbf{G}}(\mathrm{bl}(\zeta_0)) = \mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\mathrm{bl}(\zeta))$?

F.–Malle–Zhang: This holds for unipotent blocks!

Happy birthday, Michel!